SOME VECTOR-VALUED LAPLACE TRANSFORMS

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ABSTRACT

Representation theorems for vector-valued Laplace transforms are discussed. Necessary and sufficient conditions are obtained in order that a function be the Laplace transform of a general vector measure and of a vector measure of finite variation, finite q-variation or finite q-semi-variation for $1 < q \leqq \infty$.

1. Introduction

Let X be a Banach space and let $f(s)$ denote a function defined on $[0, \infty)$ with values in X. We shall be interested in finding conditions, necessary and sufficient, in order that $f(s)$ be the Laplace integral (with respect to some vector measure defined on the Borel subsets of $[0, \infty)$) which assumes values in X or in X^{**}, that is,

$$f(s) = \int_0^\infty e^{-st}\mu(dt) \text{ for } s > 0, \tag{1}$$

where the integral on the right-hand side of (1) exists in one sense or another. We shall prescribe conditions on our vector measure such as finite variation or semi-variation or finite q-variation or q-semi-variation, for $1 < q \leqq \infty$, and obtain necessary and sufficient conditions for the representation to hold.

In recent years extensive work has been done on this subject by Miyadera [5] and Zaidman [9]–[13]. Most recently, in his thesis [8], Whitford has made a substantial contribution to the theory. While all of the above papers have made use of the inverse operator of Widder [7], it is our purpose here to obtain necessary and sufficient conditions of a different type. We also give a complete solution to problems in cases that have been left open by the authors mentioned above.

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In Section 2 we discuss measures of finite variation or semivariation and in Section 3 we deal with measures of finite q-variation or q-semi-variation.

2. Measures of finite variation or semi-variation

Let μ be a measure defined on the Borel subsets of $[0, \infty)$ with values in a Banach space X. Define the semi-variation of μ by

$$\| \mu \| [0, \infty) = \sup \left\| \sum_{i=1}^{n} \alpha_i \mu A_i \right\|$$

where the supremum is taken over all finite collections of disjoint Borel sets in $[0, \infty)$ and scalars α_i with $|\alpha_i| \leqq 1$, and all $n \geqq 1$. The semi-variation of a vector measure is always finite. Define the variation of μ by

$$|||\, \mu \,||| [0, \infty) = \sup \sum_{i=1}^{n} \| \mu A_i \|$$

where the supremum is taken over all finite collections of disjoint Borel sets in $[0, \infty)$. Evidently $\| \mu \| [0, \infty) \leqq |||\, \mu \,||| [0, \infty)$ and the variation of μ need not be finite.

Our first result extends a result of Widder [7] in the case where X is the complex field; but employs a different method of proof.

THEOREM 1. *Let $f(s)$ be a vector-valued function from $[0, \infty)$ into X. Then there exists a vector measure defined on the Borel sets in $[0, \infty)$ with values in X^{**} such that*

(2) $\mu(\cdot)x^*$ *is in* rca$[0, \infty)$ *for each* $x^* \in X^*$;

(3) *the mapping* $x^* \to \mu(\cdot)x^*$ *is continuous in the* w^* *topologies of* X^* *and* rca$[0, \infty)$;

(4) $$x^*f(s) = \int_0^\infty e^{-st}\mu(dt)x^* \text{ for } s > 0 \text{ with } x^* \in X^*;$$

if and only if $f(s)$ is differentiable infinitely often in $(0, \infty)$ and

(5) $$\sup \left\| \sum_{k=0}^{n} \alpha_k \frac{s^k}{k!} f^{(k)}(s) \right\| \equiv H < \infty$$

where the supremum is taken over all finite sets of scalars α_i with $|\alpha_i| \leqq 1$, all $n \geqq 0$, and all $s > 0$. Moreover

(6) $$\| \mu \| [0, \infty) = H.$$

PROOF. Suppose μ exists such that (2), (3), and (4) hold; then for each $x^* \in X^*$ $x^*f(s)$ is holomorphic in $s > 0$, hence $f(s)$ is differentiable infinitely often. By (4)

$$x^* \left[\sum_{k=0}^{n} \alpha_k \frac{s^k}{k!} f^{(k)}(s) \right] = \int_0^\infty \left[\sum_{k=0}^{n} \alpha_k \frac{(-st)^k}{k!} e^{-st} \right] \mu(dt) x^*$$

whence

$$\sup \left\| \sum_{k=0}^{n} \alpha_k \frac{s^k}{k!} f^{(k)}(s) \right\| \leqq \| \mu \| [0, \infty). \tag{7}$$

Conversely, suppose that $f(s)$ is differentiable infinitely often in $(0, \infty)$ and that (5) holds. Then a crucial part in the proof (and in fact throughout the whole paper is the observation that (5) implies that for every $\lambda > 0$,

(8)

$$\sum_{k=0}^{\infty} e^{-\lambda k/s} \frac{(-s)^k}{k!} f^{(k)}(s) \text{ exists, and that}$$

$$\lim_{s \to \infty} \sum_{k=0}^{\infty} e^{-\lambda k/s} \frac{(-s)^k}{k!} f^{(k)}(s) = f(\lambda), \ \lambda > 0.$$

In order to prove this let $x^* \in X^*$. Then by (5)

$$\left\| \sum_{k=0}^{n} \alpha_k \frac{s^k}{k!} x^* f^{(k)}(s) \right\| \leqq H \| x^* \|,$$

for all $n \geqq 0$ and all finite sets of α_i with $|\alpha_i| \leqq 1$. Hence

$$\sum_{k=0}^{\infty} \frac{s^k}{k!} | x^* f^{(k)}(s) | \leqq H \| x^* \|.$$

In turn this implies that $x^*f(s)$ is holomorphic in $s > 0$ (see Widder [7, Th. VII.13]). Consequently

$$\sum_{k=0}^{\infty} e^{-\lambda k/s} \frac{(-s)^k}{k!} f^{(k)}(s) = f^{(k)}(s(1 - e^{-\lambda/s}))$$

(see [3, Secs. 3.10, 3.11]) and (8) is evident.

Define now the continuous operator $U: C_0[0, \infty) \to X$ (where $C_0[0, \infty)$ is the space of all continuous functions in $[0, \infty)$ which vanish at infinity, normed by the sup-norm) as follows. Since for each $\lambda > 0$, $e^{-\lambda t} \in C_0[0, \infty)$, we define $U(e^{-\lambda t}) = f(\lambda)$ and extend U linearly to the linear combinations of the functions $\{e^{-\lambda t}\}$, $\lambda > 0$. Now if

$$g(t) = \sum_{i=1}^{m} a_i \exp(-\lambda_i t),$$

then by (5) and the uniform boundedness principle it follows that

$$\left\| \sum_{k=0}^{\infty} g\left(\frac{k}{s}\right) \frac{(-s)^k}{k!} f^{(k)}(s) \right\| \leqq H \sup_{0 \leqq t < \infty} |g(t)|.$$

This in turn implies by (8) that

$$(9) \qquad \begin{aligned} \| Ug \| &= \lim_{s \to \infty} \left\| \sum_{k=0}^{\infty} g\left(\frac{k}{s}\right) \frac{(-s)^k}{k!} f^{(k}(s) \right\| \\ &\leqq H \| g \|. \end{aligned}$$

Since the linear combinations of $\{e^{-\lambda t}\}$, $\lambda > 0$, are dense in $C_0[0, \infty)$ it follows by (9) that U may be extended to a continuous operator on $C_0[0, \infty)$ which we will continue to call U. Now there exists a vector measure satisfying (2) and (3) (similar to the Bartle-Dunford-Schwartz theorem [2, Th. VI.7.2]) such that for all $g \in C_0[0, \infty)$ and all $x^* \in X^*$

$$x^* Ug = \int_0^{\infty} g(t) \mu(dt) x^*.$$

In particular we obtain (4) by taking $g(t) = e^{-st}$ for $s > 0$. Also $\| \mu \| [0, \infty) = \| U \| \leqq H$ by (9); hence (6) follows by (7). This completes the proof.

Corollary 2 is an immediate consequence of Theorem 1.

Corollary 2. *If X is reflexive and $f(s)$ is a vector-valued function from $[0, \infty)$ into X, then there exists a measure, defined on the Borel subsets of $[0, \infty)$ into X, such that*

$$(10) \qquad x^* \mu(\cdot) \text{ is in } \mathrm{rca}[0, \infty) \text{ for each } x^* \in X^*;$$

$$(11) \qquad f(s) = \int_0^{\infty} e^{-st} \mu(dt) \text{ for } s > 0,$$

if and only if $f(s)$ is differentiable infinitely often and (5) *holds.*

We may want μ to take values in X without assuming that X is reflexive. Theorem 3 achieves this by strengthening condition (5).

Theorem 3. *Let $f(s)$ be a vector-valued function from $[0, \infty)$ into X. Then there exists a measure, defined on the Borel subsets of $[0, \infty)$ into X, such that* (10) *and* (11) *hold if and only if $f(s)$ is differentiable infinitely often in* $(0, \infty)$ *and the set*

$$\left\{\sum_{k=0}^{n} \alpha_k \frac{s^k}{k!} f^{(k)}(s) : |\alpha_k| \leqq 1, 0 \leqq k \leqq n = 0,1,2,\cdots\right\} \tag{12}$$

is relatively weakly compact.

Another characterization may be found in [8, Th. 1.62].

PROOF. First assume that (10) and (11) are satisfied. Then

$$\sum_{k=0}^{n} \alpha_k \frac{s^k}{k!} f^{(k)}(s) = \int_0^\infty \left[\sum_{k=0}^{n} \alpha_k \frac{(-st)^k}{k!} e^{-st}\right] \mu(dt). \tag{13}$$

Now for any $n \geqq 0$ and $|\alpha_k| \leqq 1$, $0 \leqq k \leqq n$,

$$\left|\sum_{k=0}^{n} \alpha_k \frac{(-st)^k}{k!} e^{-st}\right| \leqq e^{-st} \sum_{k=0}^{\infty} \frac{(st)^k}{k!} = 1. \tag{14}$$

Therefore the set (12) is contained in the absolute closed convex hull of $R(\mu)$, the range of μ. By [6], $R(\mu)$ is relatively weakly compact; consequently, by the Krein-Šmulian theorem (see [2, Th. V.6.4]), the absolute closed convex hull of $R(\mu)$ is weakly compact, hence (12) is relatively weakly compact.

Conversely, suppose (12) is relatively weakly compact. Then (5) holds and we may define the operator U of the proof of Theorem 1. The operator U is weakly compact. For if we have a bounded sequence of continuous functions $\{g_i(t)\}$, $|g_i| \leqq M$ say, then the sequence $\{Ug_i\}$ is contained in the closure of the set

$$\left\{\sum_{k=0}^{n} \alpha_k \frac{s^k}{k!} f^{(k)}(s) : |\alpha_k| \leqq M,\ 0 \leqq k \leqq n = 0,1,2,\cdots\right\}$$

which is weakly compact. Now there exists a vector measure from the Borel subsets of $[0,\infty)$ into X similar to [2, Th. VI.7.3] which satisfies (10) such that for all $g \in C_0[0,\infty)$

$$Ug = \int_0^\infty g(t)\mu(dt).$$

In particular, for $g(t) = e^{-st}$ where $s > 0$, we obtain (11). This completes the proof.

We proceed now to vector measures of finite variation. One result is Theorem 4.

THEOREM 4. *Let $f(s)$ be a vector-valued function from $[0,\infty)$ into X. Then there exists a regular vector measure of finite variation, defined on the Borel subset of $[0,\infty)$ into X, such that* (11) *holds if and only if $f(s)$ is differentiable infinitely often in $(0,\infty)$ and*

$$\sup_{s>0} \sum_{k=0}^{\infty} \left\|\frac{s^k}{k!} f^{(k)}(s)\right\| \equiv H < \infty. \tag{15}$$

Furthermore $|||\mu|||\,[0,\infty) = H$.

For a different condition see [8, Th. 1.10].

PROOF. As before, (11) implies that $f(s)$ has derivatives of all orders, and that (15) is necessary follows immediately by (13) and (14) which also imply that $H \leqq |||\mu|||\,[0,\infty)$.

Conversely, suppose $f(s)$ has derivatives of all orders and that (15) holds. Since (5) implies

$$\lim_{s\to\infty} \sum_{k=0}^{\infty} e^{-\lambda k/s}\frac{(-s)^k}{k!}f^{(k)}(s) = f(\lambda),$$

this is certainly true with (15) replacing (5). Now the linear combinations of $\{e^{-\lambda t}\}$ for $\lambda > 0$ are dense in $C_0[0,\infty)$, therefore it follows easily by (15) that for each $g \in C_0[0,\infty)$

$$\lim_{s\to\infty} \sum_{k=0}^{\infty} g\left(\frac{k}{s}\right)\frac{(-s)^k}{k!}f^{(k)}(s) \text{ exists.}$$

The operator U of the proof of Theorem 1 is thus given by

$$Ug = \lim_{s\to\infty} \sum_{k=0}^{\infty} g\left(\frac{k}{s}\right)\frac{(-s)^k}{k!}f^{(k)}(s). \tag{16}$$

We prove now that for any finite set of functions $g_1,\cdots,g_n \in C_0[0,\infty)$ with $\sum_{i=1}^{n}|g_i| \leqq 1$ we have

$$\sum_{i=1}^{n} \|Ug_i\| \leqq H. \tag{17}$$

Then it follows, similar to [1, Th. 19.2, 19.3] (see also [8, Lem. 0.6]), that there exists a regular vector measure of finite variation, defined on the Borel subsets of $[0,\infty)$ into X, such that

$$Ug = \int_0^\infty g(t)\mu(dt) \text{ for } g \in C_0[0,\infty).$$

In particular (11) follows for $g(t) = e^{-st}$ where $s > 0$. Thus let $g_1,\cdots,g_n \in C_0[0,\infty)$ with $\sum_{i=1}^{n}|g_i| \leqq 1$ and let us prove (17). By (16)

$$\begin{aligned}\|Ug_i\| &\leqq \lim_{s\to\infty}\left\|\sum_{k=0}^{\infty} g_i\left(\frac{k}{s}\right)\frac{(-s)^k}{k!}f^{(k)}(s)\right\| \\ &\leqq \lim_{s\to\infty}\sum_{k=0}^{\infty}\left|g_i\left(\frac{k}{s}\right)\right|\left\|\frac{s^k}{k!}f^{(k)}(s)\right\|,\end{aligned}$$

where the last limit exists by (15). Hence

$$\sum_{i=1}^{n} \| Ug_i \| \leqq \lim_{s \to \infty} \sum_{k=0}^{\infty} \sum_{i=0}^{n} \left| g_i\left(\frac{k}{s}\right) \right| \left\| \frac{s^k}{k!} f^{(k)}(s) \right\| \leqq H.$$

This also implies $|||\mu|||\,[0,\infty) \leqq H$ which completes the proof.

3. Measures of finite p-variation or p-semi-variation

Let μ be a vector measure, defined on the Borel subsets of $[0,\infty)$ into X. Let $1 \leqq p \leqq \infty$ and $1/p + 1/q = 1$. Define the p-semi-variation of μ as follows (see [1, p. 246]).

$$\tilde{\mu}_p[0,\infty) = \sup \left\| \sum_{=1}^{n} \alpha_i \mu A_i \right\|$$

where the supremum is taken on all step functions $\phi = \sum_{i=1}^{n} \alpha_i \chi_{A_i}$ such that the A_i's are pairwise disjoint Borel subsets of $[0,\infty)$ and such that

$$\int_0^\infty |\phi(t)|^q dt \leqq 1.$$

Also define the p-variation of μ as follows (see [1, p. 241]).

$$\bar{\mu}_p[0,\infty) = \sup \sum_{i=1}^{n} |\alpha_i| \; \| \mu A_i \|$$

where the supremum is taken on all the step function ϕ mentioned above.

Our results in this section generalize representation theorems for Laplace transforms of functions in $L^p[0,\infty)$. Our conditions are similar to those in Section 2, and therefore different from the well-known conditions of Widder [7] which, in the case of reflexive X, were generalized by Miyadera [5].

First we prove Theorem 5.

THEOREM 5. *Let $f(s)$ be a vector-valued function from $[0,\infty)$ into X, and let $1 < p \leqq \infty$. Then there is a measure μ, defined on the Borel subsets of $[0,\infty)$ into X, with finite p-semi-variation such that* (11) *holds if and only if $f(s)$ is differentiable infinitely often in* $(0,\infty)$ *and*

$$\text{(18)} \qquad \sup_{|x^*| \leqq 1} \sup_{s>0} \sum_{k=0}^{\infty} s^{p-1} \left| \frac{s^k}{k!} x^* f^{(k)}(s) \right|^p \equiv (H_p)^p < \infty \text{ if } 1 < p < \infty$$

and

$$\text{(19)} \qquad \sup_{s>0} \frac{s^{k+1}}{k!} \| f^{(k)}(s) \| \equiv H_\infty < \infty \text{ if } p = \infty.$$

Furthermore $\tilde{\mu}_p[0,\infty) = H_p$, $1 < p \leqq \infty$.

PROOF. Suppose first that (11) holds with a vector measure μ such that $\tilde{\mu}_p[0,\infty) < \infty$ for some $1<p\leqq\infty$. Then it follows (by [1, Prop. 13.1] and a well-known Riesz theorem) that for each fixed $x^* \in X^*$ with $\| x^* \| \leqq 1$, the measure $x^*\mu(\cdot)$ is the indefinite integral of a function in $L^p[0,\infty)$, that is, $\phi \in L^p[0,\infty)$ such that $x^*\mu A = \int_A \phi(t)\,dt$ and

$$(20) \qquad \int_0^\infty |\phi(t)|^p dt = (\widetilde{x^*\mu})_p[0,\infty) \leqq \tilde{\mu}_p[0,\infty) \text{ if } 1 < p < \infty$$

or

$$(21) \qquad \operatorname*{ess\,sup}_{s>0} |\phi(t)| = (\widetilde{x^*\mu})_\infty[0,\infty) \leqq \tilde{\mu}_\infty[0,\infty) \text{ if } p = \infty.$$

Also it follows by (11) that

$$x^*f(s) = \int_0^\infty e^{-st}\phi(t)dt \text{ for } s > 0.$$

Hence it follows by [4, Th. 1, 2] and (20) (if $1 < p < \infty$) and by [7, Th. VII.16a] and (21) (if $p = \infty$) that for each fixed $\| x^* \| \leqq 1$, $x^*f(s)$ has derivatives of all orders in $(0,\infty)$. This implies in turn (see [3]) that the same is true for $f(s)$, and

$$(22) \qquad \sup_{s>0} \sum_{k=0}^\infty s^{p-1} \left| \frac{s^k}{k!} x^*f^{(k)}(s) \right|^p \leqq \int_0^\infty |\phi(t)|^p dt$$
$$\leqq \tilde{\mu}_p[0,\infty) \text{ if } 1 < p < \infty,$$

or

$$(23) \qquad \sup_{s>0} \frac{s^{k+1}}{k!} |x^*f^{(k)}(s)| \leqq \operatorname*{ess\,sup}_{s>0} |\phi(t)|$$
$$\leqq \tilde{\mu}_\infty[0,\infty) \text{ if } p = \infty.$$

This in turn implies (18) and (19) respectively. Conversely, suppose either that (18) holds (if $1 < p < \infty$) or (19) holds (if $p = \infty$). In the case where $1 < p < \infty$, then (18) implies that $x^*f(s)$ is holomorphic in $(0,\infty)$ (by [4, Proof of Th. 1]). The same method of proof may be used to show that if $p = \infty$, then (19) implies the same. Therefore (8) holds in either cases (see [3]). Now the functions $\{e^{-\lambda t}\}$ for $\lambda > 0$ are all in $L^q[0,\infty)$ for every $1 \leqq q < \infty$ and their linear combinations are dense in $L^q[0,\infty)$ for $1 \leqq q < \infty$ in the respective norms. Therefore if we define

$$U(e^{-\lambda t}) = f(\lambda)$$

and extend it linearly to the linear combinations of $\{e^{-\lambda t}\}$ for $\lambda > 0$, then we will show that U may be extended to a continuous operator (again denoted by U) of $L^q(0,\infty)$ into X where $1/p + 1/q = 1$. To this end let $g(t) = \sum_{i=1}^{n} a_i \exp(-\lambda_i t)$. Then by (8)

$$Ug = \lim_{s\to\infty} \sum_{k=0}^{\infty} g\left(\frac{k}{s}\right) \frac{(-s)^k}{k!} f^{(k)}(s). \tag{24}$$

Now let $1 < p < \infty$; then for any fixed $\| x^* \| \leqq 1$ we have by Hölder's inequality and (18)

$$\begin{aligned} |x^*Ug| &\leqq \limsup_{s\to\infty} \sum_{k=0}^{\infty} \left| g\left(\frac{k}{s}\right)\right| \left|\frac{s^k}{k!} x^* f^{(k)}(s)\right| \\ &\leqq \limsup_{s\to\infty} \left(\sum_{k=0}^{\infty} \left| g\left(\frac{k}{s}\right)\right|^q \frac{1}{s}\right)^{1/q} \left(\sum_{k=0}^{\infty} s^{p-1} \left|\frac{s^k}{k!} x^* f^{(k)}(s)\right|^p\right)^{1/p} \\ &\leqq \lim_{s\to\infty} \left(\sum_{k=0}^{\infty} \left| g\left(\frac{k}{s}\right)\right|^q \frac{1}{s}\right)^{1/q} H_p. \end{aligned} \tag{25}$$

Now it is readily seen that for each linear combination of the $\{e^{-\lambda t}\}$ $\lambda > 0$,

$$\lim_{s\to\infty} \left(\sum_{k=0}^{\infty} \left| g\left(\frac{k}{s}\right)\right|^q \frac{1}{s}\right)^{1/q} = \left(\int_0^\infty |g(t)|^q dt\right)^{1/q}$$

so that (25) implies

$$\| Ug \| \leqq H_p \| g \|_q. \tag{26}$$

Now that U is defined for all functions in $L^q[0,\infty)$ into X it follows by (26) that for all $g \in L^q[0,\infty)$ inequality (26) holds. Since $\| U \|_q \leqq H_p$, then it follows (by [1, Th. 13.1]) that there exists a vector measure μ with finite p-semi-variation such that for all $g \in L^q[0,\infty)$

$$Ug = \int_0^\infty g(t)\mu(dt).$$

Specializing to $g(t) = e^{-st}$ for $s > 0$ we obtain (11). Also by [1, Th. 13.1], $\tilde{\mu}_p[0,\infty) = \| U \|_q \leqq H_p$ which, together with (22), implies $\tilde{\mu}_p[0,\infty) = H_p$. This completes the proof for $1 < p < \infty$. For $p = \infty$, it follows by (19) that for

$$g(t) = \sum_{i=1}^{n} a_i \exp(-\lambda_i t)$$

$$\begin{aligned} \| Ug \| &\leqq H_\infty \lim_{s\to\infty} \sum_{k=0}^{\infty} \left| g\left(\frac{k}{s}\right)\right| \frac{1}{s} \\ &= H_\infty \| g \|_1. \end{aligned}$$

The proof is concluded now as in the case $1 < p < \infty$.

Since for $p = \infty$, $\tilde{\mu}_p[0,\infty) = \bar{\mu}_p[0,\infty)$, Theorem 5 also provides us with a representation theorem for measures of finite ∞-variation. In a way this explains why (19) resembles condition (27) below more than it does condition (18).

For measures with finite p-variation we have the following result.

THEOREM 6. *Let $f(s)$ be a vector-valued function from $[0,\infty)$ into X and let $1 < p < \infty$. Then there exists a vector measure μ, defined on the Borel subsets of $[0,\infty)$ into X, of finite p-variation and such that (11) holds if and only if $f(s)$ has derivatives of all orders in $(0,\infty)$ and*

$$\sup_{s>0} \sum_{k=0}^{\infty} s^{p-1}\left\|\frac{s^k}{k!}f^{(k)}(s)\right\|^p \equiv (H_p)^p < \infty. \tag{27}$$

Furthermore

$$\bar{\mu}_p[0,\infty) = H_p.$$

PROOF. First suppose (11) holds with μ of finite p-variation. Then

$$\frac{(-s)^k}{k!}f^{(k)}(s) = \int_0^\infty \frac{(st)^k}{k!}e^{-st}\mu(dt) \text{ for } s > 0.$$

Denote

$$p_{sk}(t) = \frac{(st)^k}{k!}\, e^{-st} \text{ for } s > 0,\ k = 0,1,2,\cdots$$

and let for $\lambda > 0$, $A_i = [i/\lambda, (i+1)/\lambda)$, $i = 0,1,2,\cdots$. Then the step function $\phi_k = \sum_{i=0}^{\infty} p_{sk}(i/\lambda)\chi_{A_i}$ is defined in $[0,\infty)$ and $\phi_k(t) \to p_{sk}(t)$ as λ approaches infinity in the $L^q[0,\infty)$ norm for every $1 \leqq q \leqq \infty$. Consequently for each fixed k

$$\int_0^\infty \phi_k(t)\mu(dt) \to \frac{(-s)^k}{k!}f^{(k)}(s) \text{ as } \lambda \to \infty \tag{28}$$

and

$$\int_0^\infty \phi_k(t)dt \to \frac{1}{s} \text{ as } \lambda \to \infty. \tag{29}$$

Now let n be arbitrary, then

$$\sum_{k=0}^{n} \left\{\left\|\int_0^\infty \phi_k(t)\mu(dt)\right\|^p \Big/ \left(\int_0^\infty \phi_k(t)dt\right)^{p-1}\right\}$$
$$\leqq \lambda^{p-1}\sum_{k=0}^{n}\left\{\left(\sum_{i=0}^{\infty} p_{sk}(i/\lambda)\left\|\mu A_i\right\|^p\right) \Big/ \left(\sum_{i=0}^{\infty} p_{sk}(i/\lambda)\right)^{p-1}\right\}$$

By Jensen's inequality (see [14, p. 23–24]) the last expression is

$$\leqq \lambda^{p-1} \sum_{k=0}^{n} \left(\sum_{i=0}^{\infty} p_{sk}(i/\lambda) \right) \left(\sum_{i=0}^{\infty} p_{sk}(i/\lambda) \| \mu A_i \|^p \right) \Big/ \left(\sum_{i=0}^{\infty} p_{sk}(i/\lambda) \right)$$

$$= \lambda^{p-1} \sum_{k=0}^{n} \sum_{i=0}^{\infty} p_{sk}(i/\lambda) \| \mu A_i \|^p$$

$$= \lambda^{p-1} \sum_{i=0}^{\infty} \| \mu A_i \|^p \sum_{k=0}^{n} p_{sk}(i/\lambda).$$

Since $\sum_{k=0}^{\infty} p_{sk}(t) = 1$ for $0 \leqq t < \infty$ and $p_{sk}(t) \geqq 0$, it follows now by [1, Prop. 13.1] that

$$\sum_{k=0}^{n} \left\{ \left\| \int_0^\infty \phi_k(t) \mu(dt) \right\|^p \Big/ \left(\int_0^\infty \phi_k(t) dt \right)^{p-1} \right\} \leqq \sum_{i=0}^{\infty} \left\{ \| \mu A_i \|^p \Big/ \left(\frac{1}{\lambda} \right)^{p-1} \right\}$$

$$\leqq (\bar{\mu}_p[0, \infty))^p.$$

Letting λ approach infinity we obtain by (28) and (29),

$$\sum_{k=0}^{n} s^{p-1} \left\| \frac{s^k}{k!} f^{(k)}(s) \right\| \leqq (\bar{\mu}_p[0, \infty))^p$$

which, being true for all n and all s, implies (27) and

$$H_p \leqq \bar{\mu}_p[0, \infty). \tag{30}$$

Conversely, suppose that $f(s)$ has derivatives of all orders and that (27) holds. Then evidently (18) holds. Hence the operator U of the proof of Theorem 5 may be defined. This time we know more about the operator U. If A is a set of finite Lebesgue measure then $\chi_A \in L^q[0, \infty)$. It follows by (26) that

$$U\chi_A = \lim_{s \to \infty} \sum_{k=0}^{\infty} \chi_A\left(\frac{k}{s}\right) \frac{(-s)^k}{k!} f^{(k)}(s). \tag{31}$$

For the linear combinations of the functions $\{e^{-\lambda t}\}$ for $\lambda > 0$ are dense in $L^q[0, \infty)$ and we have for such a combination $g(t)$,

$$\left\| \sum_{k=0}^{\infty} \chi_A\left(\frac{k}{s_1}\right) \frac{(-s_1)^k}{k!} f^{(k)}(s_1) - \sum_{k=0}^{\infty} \chi_A\left(\frac{k}{s_2}\right) - \frac{(-s_2)^k}{k!} f^{(k)}(s_2) \right\|$$

$$\leqq \left\| \sum_{k=0}^{\infty} \left[\chi_A\left(\frac{k}{s_1}\right) - g\left(\frac{k}{s_1}\right) \right] \frac{(-s_1)^k}{k!} f^{(k)}(s_1) \right\| +$$

$$+ \left\| \sum_{k=0}^{\infty} g\left(\frac{k}{s_1}\right) \frac{(-s_1)^k}{k!} f^{(k)}(s_1) - \sum_{k=0}^{\infty} g\left(\frac{k}{s_2}\right) \frac{(-s_2)^k}{k!} f^{(k)}(s_2) \right\| +$$

$$+ \left\| \sum_{k=0}^{\infty} \left[\chi_A\left(\frac{k}{s_2}\right) - g\left(\frac{k}{s_2}\right) \right] \frac{(-s_2)^k}{k!} f^{(k)}(s_2) \right\|$$

$$= I_1 + I_2 + I_3, \text{ say.}$$

Now by Hölder's inequality (compare with (25))

$$I_1 \leqq \sum_{k=0}^{\infty} \left|\chi_A\left(\frac{k}{s_1}\right) - g\left(\frac{k}{s_1}\right)\right| \quad \left\|\frac{s_1^k}{k!} f^{(k)}(s_1)\right\|$$

$$\leqq \left(\sum_{k=0}^{\infty} \left|\chi_A\left(\frac{k}{s_1}\right) - g\left(\frac{k}{s_1}\right)\right|^q \frac{1}{s}\right)^{1/q} H_p$$

$$\to H_p \int_0^\infty |\chi_A(t) - g(t)|^q dt,$$

since χ_A is the characteristic function of a set of finite Lebesgue measures and $g(t)$ is a linear combination of the $\{e^{-\lambda t}\}$ for $\lambda > 0$. Hence for a proper $g(t)$ and all s, sufficiently large $I_1 \leqq \varepsilon$. The same is true for I_3 while by (24) $I_2 \to 0$ as $s_1, s_2 \to \infty$ This proves (31).

Let $A_1, \cdots, A_n$ be disjoint Borel subsets of $[0, \infty)$, each of finite measure, and let $\alpha_1, \cdots, \alpha_n$ be constants. Then by (31)

$$\sum_{i=1}^{n} \| U(\alpha_i \chi_{A_i}) \| \leqq \limsup_{s \to \infty} \sum_{k=0}^{\infty} \sum_{i=1}^{n} \left|\alpha_i \chi_{A_i}\left(\frac{k}{s}\right)\right| \quad \left\|\frac{s^k}{k!} f^{(k)}(s)\right\|$$

$$\leqq \lim_{s \to \infty} \left(\sum_{k=0}^{\infty} \left(\sum_{i=1}^{n} \left|\alpha_i \chi_A\left(\frac{k}{s}\right)\right|\right)^q \frac{1}{s}\right)^{1/q} H_p$$

$$= \left(\int_0^\infty | \phi(t)|^q dt\right)^{1/q} H_p,$$

where $\phi = \sum_{i=1}^n \alpha_i \chi_{A_i}$.

Therefore if we define the norm

$$||| U |||_q = \sup \sum_{i=1}^{n} \| U(\alpha_i \chi_{A_i}) \|$$

where the supremum is taken on all step functions $\phi = \sum_{i=1}^n \alpha_i \chi_{A_i}$ (where $A_1, \cdots, A_n$ are pairwise disjoint and $\|\phi\|_q \leqq 1$), then

$$(32) \qquad ||| U |||_q \leqq H_p.$$

This in turn implies by [1, Th. 13.1 and Cor. 13.1] that there exists a vector measure μ, defined on the Borel subsets of $[0, \infty)$ into X, such that

$$\bar{\mu}_p[0, \infty) = ||| U |||_a$$

and for all $g \in L^q[0, \infty)$

$$Ug = \int_0^\infty g(t) \mu(dt).$$

Specializing to $g(t) = e^{-st}$ for $s > 0$ we obtain (11). Combining now (30) and (32) we conclude that $\bar{\mu}_p[0, \infty) = H_p$. This completes the proof.

REMARK. If X is the complex field, then $\tilde{\mu}_p[0, \infty)$ and $\bar{\mu}_p[0, \infty)$ coincide and so do Theorems 4 and 5.

In order to relate Theorem 6 more closely to vector L^p spaces we cite our next result (whose proof is similar to that of Theorem 5) making use of [1, Th. 13.8] instead of [1, Th. 13.1 and Cor. 13.1], namely Theorem 7.

THEOREM 7. *Let $f(s)$ be a vector-valued function from $[0, \infty)$ into X, let Z be a norming subspace of X^*, and let $1 < p \leqq \infty$. Then there exists a function $G(t)$ from $[0, \infty)$ into Z^* with $\| G(t) \| \in L^p[0, \infty)$ such that for all $z \in Z$*

$$zf(s) = \int_0^\infty e^{-st} G(t) z dt \text{ for } s > 0,$$

if and only if (27) holds (for $1 < p < \infty$) and (19) holds (for $p = \infty$). Moreover

$$\left(\int_0^\infty \| G(t) \|^p dt\right)^{1/q} = H_p \text{ if } 1 < p < \infty$$

and

$$\operatorname*{ess\,sup}_{0 \leqq t < \infty} \| G(t) \| = H_\infty \text{ if } p = \infty.$$

If X is reflexive we may take $Z = X^*$ and have the following corollary as an immediate consequence.

COROLLARY 8. *Let $f(s)$ be a vector-valued function from $[0, \infty)$ into a reflxive space X and let $1 < p \leqq \infty$. Then there exists a function G from $[0, \infty)$ into X with $\| G(t) \| \in L^p[0, \infty)$ such that*

$$f(s) = \int_0^\infty e^{-st} G(t) dt \text{ for } s > 0, \tag{33}$$

if and only if (27) holds (if $1 < p < \infty$) and (19) holds (if $p = \infty$).

Another necessary and sufficient condition for (33) to hold with $G: [0, \infty) \to X$ has been given by Miyadera [5]. If X is not reflexive, neither our condition nor Miyadera's is sufficient for the existence of $G: [0, \infty) \to X$ as demonstrated by Theorem 7, in our case, and explained by Whitford [7, Rem. 1.24] about Miyadera's condition. We do not know the analog of Theorem 7 with Miyadera's condition; nevertheless Theorem 7 provides a complete picture of the non-reflexive case.

To conclude, Theorem 9 is a consequence of Theorem 7 and of [1, Th. 13.8].

THEOREM 9. *Let $f(s)$ be a vector-valued function from $[0,\infty)$ into X and let $1 < p \leqq \infty$. If (27) holds (for $1 < p < \infty$) or (19) holds (for $p = \infty$) and if, in addition, the set*

$$\left\{ \sum_{k=0}^{\infty} \phi\left(\frac{k}{s}\right) \frac{(-s)^k}{k!} f^{(k)}(s) \text{ where } \phi \text{ is a step function and } \int_0^\infty |\phi| dt \leqq 1, s > 0 \right\}$$

is relatively weakly compact, then there exists a function $G(t)$ from $[0,\infty)$ into X with $\| G(t) \| \in L^p[0.\infty)$ such that

$$f(s) = \int_0^\infty e^{-st} G(t)\, dt \quad \text{for } s > 0.$$

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